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Derivation of low-temperature series expansions for the Ising model with triplet interactions on the plane triangular lattice

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Abstract. The derivation of low-temperature (high-field) series expansions for the Ising model with pure triplet interactions on the plane triangular lattice is described. Euler's law of the edges is used to transform the linkage rule into a form convenient for the derivation of ferromagnetic polynomials. Explicit results are given for the ferromagnetic polynomials corresponding to the first twelve powers of the temperature variable (u_3) both as functions of the field variable (μ) and, in zero field, as functions of the temperature variable (u_2) of the simple Ising model.

1. Introduction

Ising models with multi-spin interactions have been studied by many authors (Wegner 1971, 1972, Merlini and Gruber 1972, Hintermann and Merlini 1972, Thibaudier and Villain 1972, Baxter 1974, Wood and Griffiths 1973, 1974, Merlini 1973, Merlini *et al* 1973, Griffiths and Wood 1973, Gruber *et al* 1973). In this paper we investigate the configurational problem that arises in the derivation of low-temperature and high-field expansions for the Ising model of a ferromagnet on the triangular lattice with three-spin interactions (pure triplet model). Recently the free energy of this model in the absence of a field has been solved exactly (Baxter and Wu 1973, 1974, Baxter 1974) and the spontaneous magnetization has been conjectured from a study of its series expansion (Baxter *et al* 1975). We describe the configurational background to this latter study and derive the data there used. Ferromagnetic polynomials also provide data for the investigation of the low-temperature susceptibility (Watts 1974)† and higher-field derivatives of the model.

2. The configurational problem: the linkage rule

The *simple* Ising model for a system of spins on the triangular lattice is defined by the Hamiltonian

$$\mathscr{H} = -mH\sum_i \sigma_i - J_2 \sum_{i,j} \sigma_i\sigma_j \tag{2.1}$$

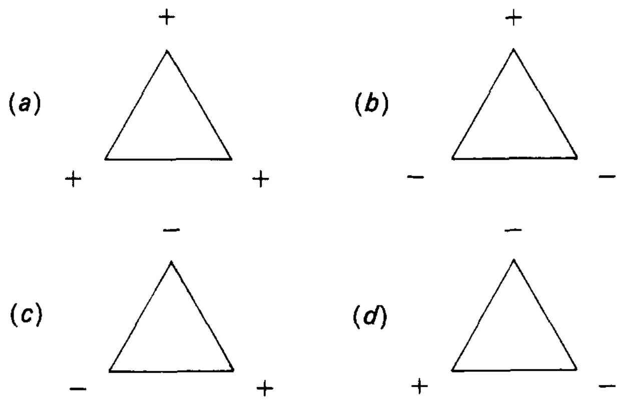
where m denotes the magnetic moment per spin, H the applied magnetic field, J_2 the pair interaction energy, and the σ_i take the conventional values ± 1. The first summation is taken over all N sites of the lattice; the second over all $3N$ bonds.

† The coefficients of u^{12} for I and χ quoted in Watts (1974) are in error by insignificant amounts.

The *pure triplet* model is defined by the Hamiltonian

$$\mathscr{H} = -mH\sum_{i}\sigma_i - J_3\sum_{i,j,k}\sigma_i\sigma_j\sigma_k \tag{2.2}$$

where J_3 denotes the triplet interaction energy and the second summation is taken over all the $2N$ elementary triangles of the lattice. To develop *high-field expansions* for the pure triplet model we study perturbations of the ordered state. In zero field there are four ground states (Merlini and Gruber 1972, Gruber *et al* 1973). For an infinite lattice the choice of any of the four arrangements:



for one triangle, together with the condition that the energy is minimal, determines the state of the whole lattice. In the presence of a field the state (*a*) is the appropriate choice and at absolute zero all the spins point one way, corresponding to a ground state energy of $-N(2J_3+mH)$. Following closely the usual treatment of the simple Ising model we write the free energy per spin F in the form

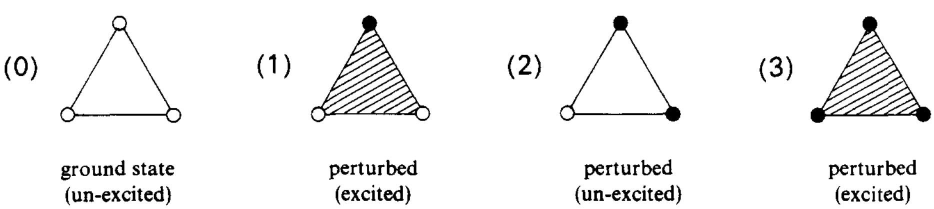
$$F = -2J_3 - mH - kT\ln\Lambda(\mu, \mu_3) \tag{2.3}$$

with

$$\begin{aligned}\mu &= \exp(-2mH/kT)\\ u_3 &= \exp(-4J_3/kT).\end{aligned} \tag{2.4}$$

The field variable μ is identical with that used for the simple Ising model; the temperature variable u_3 only differs in having J_3 in place of J_2. (For the simple model conventionally $u = u_2 = \exp(-4J_2/kT)$.)

In any perturbed state the elementary triangles can be divided into four classes characterized by the number of perturbed spins at their vertices:



(perturbed spins being denoted by full circles. Classes (1) and (2) can occur in three orientations each on the lattice.) Those of classes (0) and (2) make no contribution to the inter-spin energy above the ground state (the configurational free energy); those of classes (1) and (3), which we have shaded, contribute $2J_3$ each. We call these *excited* triangles. It is a complicating feature of the model that while the perturbation of spins in the ground state results in a corresponding perturbation of the states of the elementary triangles not all of the perturbed triangles are necessarily excited. If $[\Delta, s]$ denotes the coefficient of N (the conventional free energy count) in the total number of ways of distributing Δ excited triangles with a total of s perturbed spins then

$$\ln \Lambda = \sum_{\Delta,s} [\Delta, s] u_3^{\frac{1}{2}\Delta} \mu^s \tag{2.5}$$

the summation being taken over all possible states. The *ferromagnetic polynomials* correspond to grouping the double series (2.5) in ascending powers of u_3:

$$\ln \Lambda = \sum_i \Psi_i(\mu) u_3^i \tag{2.6}$$

and this is the form suitable for the derivation of the specific heat, spontaneous magnetization and initial susceptibility and the higher-field derivatives. We have denoted the u_3-grouping polynomials by Ψ; the u_2-grouping polynomials for the simple Ising model are usually denoted by ψ.

The whole process of series derivation is formally analogous, *mutatis mutandis*, to the corresponding theory of the simple model. A detailed treatment of the latter in the present notation is given by Sykes *et al* (1965, § 2 and 1973, § 1), see also Domb (1974).

It is readily seen from elementary geometrical considerations that while every perturbation of the spins corresponds to an arrangement of (shaded) excited triangles not every arbitrary shading of triangles on the lattice can correspond to a perturbation of spins and therefore to a valid distribution of excited triangles. It can be shown that a necessary and sufficient condition for an arrangement to be valid is that the number of excited triangles incident upon each vertex be even. To perform the summation in (2.5) it is convenient to use as parameters the number of overturned spins (s) and the number of nearest-neighbour bonds (r) and elementary triangles (t) between them. Denoting the number of triangles in each of the four classes by n_0, n_1, n_2, n_3 respectively, we have the elementary relations:

$$\begin{aligned} n_0+n_1+n_2+n_3 &= 2N \\ n_1+2n_2+3n_3 &= 6s \\ n_2+3n_3 &= 2r \\ n_3 &= t \\ \Delta = n_1+n_3 &= 6s-4r+4t. \end{aligned} \tag{2.7}$$

The summation (2.5) can now be written

$$\ln \Lambda = \sum_{s,r,t} [s, r, t] u_3^{3s-2r+2t} \mu^s = \sum_i \Psi_i u_3^i \tag{2.8}$$

where $[s, r, t]$ is the free energy count of all the perturbations that correspond to each choice of s, r and t. Perturbed spins give rise to a configurational energy given by the *linkage rule*:

$$4(3s-2r+2t)J_3+2msH \tag{2.9}$$

which may be contrasted with the corresponding linkage rule for the simple Ising model for which perturbed spins give rise to a configurational energy of (Sykes and Gaunt 1973)

$$4(3s-r)J_2+2msH. \tag{2.10}$$

In close analogy with the simple model we seek to apply the linkage rule (2.9) to the u_3-grouping (2.6); this temperature grouping provides expansions in the temperature variable u_3 for fixed values of the field variable μ. The first few ferromagnetic polynomials are readily found by inspection to be:

$$\Psi_1 = \Psi_2 = 0, \qquad \Psi_3 = \mu, \qquad \Psi_4 = 3\mu^2, \qquad \Psi_5 = 11\mu^3. \tag{2.11}$$

The temperature grouping for the simple Ising model requires a listing of configurations by ascending values of $3s-r$; the pure triplet model requires a listing by ascending values of $3s-2r+2t$. We characterize the configurations that contribute to this latter listing in the next section.

3. Application of Euler's law of the edges to the linkage rule

We have expressed the linkage rule in terms of three parameters r, s, t of the graph representing the perturbed spins and their nearest-neighbour bonds (the low-temperature configuration). We denote the number of connected components in this graph by c, and the number of finite faces by f; further we define a *hole* as a finite face which is not an elementary triangle and denote the number of these by h. By the well known result, due essentially to Euler:

$$r-s+c = t+h \tag{3.1}$$

and therefore

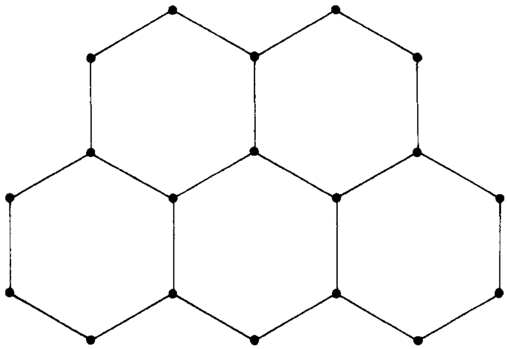
$$3s-2r+2t = s+2(c-h) = s+2\kappa. \tag{3.2}$$

We call the quantity $c-h$ the *discriminant* of the configuration and denote it by κ. It follows from (3.2) that the linkage rule can be written

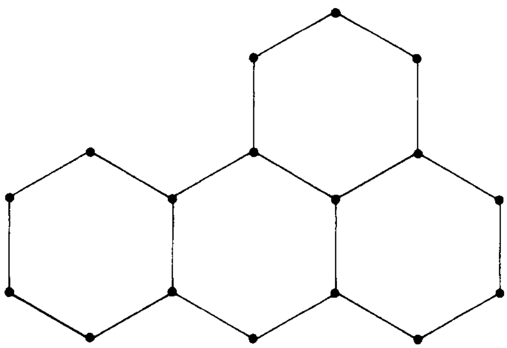
$$4(s+2\kappa)J_3+2msH \tag{3.3}$$

and the listing of configurations for a temperature grouping corresponds to a listing by ascending values of $s+2\kappa$. As the number of spins increases within a fixed power of u_3, κ must decrease; this corresponds to selecting graphs with fewer components and more holes. The listing in ascending values of s terminates with connected graphs with the maximum possible number of holes. For example, the listing for Ψ_{11} terminates at

$s = 19$ with the graph:

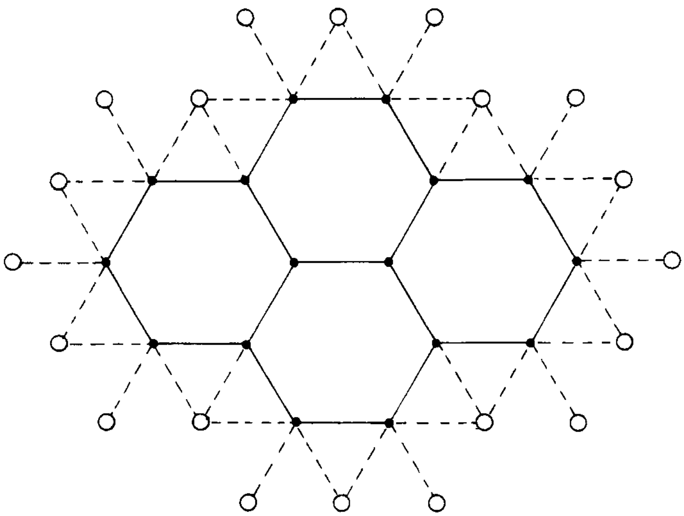


with a count of $6N$; it has five holes and a discriminant of -4. For $s = 17$ the list is based upon two arrangements of four holes:



($12N$)

Σ



($20 \times 3N$)

In the second arrangement the summation is over the 20 adjacent external sites (open circles) each of which in turn is to be occupied by a single perturbed spin. These correspond to several different topologies with $r = 20$ or 21 or 22 but this detail need not be retained since the discriminant is unaffected. This latter consideration introduces a simplification in the listing and by an exhaustive systematic study of the possible arrangements of holes we have completed the ferromagnetic polynomials through Ψ_{12}:

$$\begin{aligned}
\Psi_6 &= -3\tfrac{1}{2}\mu^2+44\mu^4+\mu^6 \\
\Psi_7 &= -30\mu^3+186\mu^5+12\mu^7 \\
\Psi_8 &= -202\tfrac{1}{2}\mu^4+813\mu^6+99\mu^8+3\mu^{10} \\
\Psi_9 &= -19\tfrac{1}{3}\mu^3-1\,250\mu^5+3\,631\mu^7+696\mu^9+51\mu^{11}+2\mu^{13} \\
\Psi_{10} &= 288\mu^4-7\,373\tfrac{1}{2}\mu^6+16\,260\mu^8+4\,473\mu^{10}+555\mu^{12}+45\mu^{14}+3\mu^{16} \\
\Psi_{11} &= 2\,889\mu^5-42\,300\mu^7+72\,994\mu^9+27\,114\mu^{11}+4\,881\mu^{13}+614\mu^{15}+72\mu^{17}+6\mu^{19} \\
\Psi_{12} &= -129\tfrac{3}{4}\mu^4+24\,301\mu^6-237\,920\mu^8+325\,066\mu^{10}+157\,512\tfrac{1}{2}\mu^{12}+37\,798\mu^{14} \\
&\quad +6\,525\mu^{16}+1\,052\mu^{18}+153\mu^{20}+14\mu^{22}+\mu^{24}.
\end{aligned} \tag{3.4}$$

Higher terms present no new difficulty of principle but would be laborious to enumerate. The polynomial Ψ_{13} terminates with $6\mu^{27}$, Ψ_{14} with $3\mu^{32}$. The values of Ψ_n ($\mu = 1$) through $n = 12$ are in agreement with the exact solution of Baxter and Wu (1974).

4. Order parameters for the triplet model

From (3.4) the spontaneous magnetization I follows from the defining relation

$$I = -\frac{1}{m}\left(\frac{\partial F}{\partial H}\right)_{H=0} = 1-2\mu\left(\frac{\partial L}{\partial u}\right)_{\mu=1} \qquad L = \ln \Lambda. \tag{4.1}$$

We obtain

$$\begin{aligned}
I &= 1-2u_3^3-12u_3^4-66u_3^5-350u_3^6-1\,848u_3^7-9\,780u_3^8-52\,012u_3^9-278\,118u_3^{10} \\
&\quad -1\,495\,092u_3^{11}-8\,077\,274u_3^{12}-\ldots.
\end{aligned} \tag{4.2}$$

Examination of the coefficients in (4.2) has lead to a conjectured exact algebraic expression for I as a function of u_3 (Baxter *et al* 1975). The critical exponent is found to be $\frac{1}{12}$ in agreement with extrapolations (Watts 1974) and the new universality hypothesis of Suzuki (1974). On the basis of the conjectured form the expansion (4.2) can be extended indefinitely; we quote the next four terms:

$$-43\,836\,468u_3^{13}-238\,889\,424u_3^{14}-1\,306\,708\,196u_3^{15}-7\,171\,779\,996u_3^{16}-\ldots. \tag{4.3}$$

The polynomials (3.4) also determine the expansions of all the higher-field derivatives; we quote the reduced initial susceptibility:

$$\begin{aligned}
\chi_0 &= u_3^3+12u_3^4+99u_3^5+726u_3^6+4\,968u_3^7+32\,664u_3^8+209\,238u_3^9+1\,316\,610u_3^{10} \\
&\quad +8\,178\,846u_3^{11}+50\,322\,488u_3^{12}+\ldots.
\end{aligned} \tag{4.4}$$

From an analysis of the coefficients of (4.4) using Padé approximants Watts (1974) concluded that the critical index $\gamma' = 1{\cdot}15 \pm 0{\cdot}15$ and that very probably $\gamma' = \frac{7}{6}$.

Baxter *et al* (1975) have conjectured the exact form of another order parameter for the pure triplet model: a bond polarization P_2 which is conveniently defined from the generalized Hamiltonian of the mixed model. Writing

$$\mathscr{H} = -mH\sum_i \sigma_i - J_2\sum_{i,j}\sigma_i\sigma_j - J_3\sum_{i,j,k}\sigma_i\sigma_j\sigma_k \tag{4.5}$$

we can define

$$P_2 = -\frac{1}{3}\left(\frac{\partial F}{\partial J_2}\right)_{J_2=0} = 1-\frac{4}{3}u_2\left(\frac{\partial F}{\partial u_2}\right)_{u_2=0} \tag{4.6}$$

It is of course not essential to introduce the mixed model to define P_2; in the present context the energy J_2 is used only as a dummy variable effectively labelling the polarity of pairs of adjacent spins. To evaluate the expansion coefficients in zero field we require the ferromagnetic polynomials $\Psi_n(\mu)$ with a modified argument: $\Psi_n(u_2)$ corresponding to the zero-field ($\mu = 1$) expansion:

$$\ln \Lambda = \sum_i \Psi_i(u_2)u_3^i. \tag{4.7}$$

This requires a detailed analysis of the configurational listing using both linkage rules (2.9) and (2.10). We give the values of $\Psi_n(u_2)$ through $n = 12$ in the appendix. From these and (4.6) we obtain:

$$P_2 = 1-4u_3^3-20u_3^4-100u_3^5-492u_3^6-2\,464u_3^7-12\,532u_3^8-64\,640u_3^9$$
$$-337\,340u_3^{10}-1\,777\,888u_3^{11}-9\,448\,112u_3^{12}-\ldots. \tag{4.8}$$

An exact expression for P_2 as an algebraic function of u_3 has been conjectured by Baxter *et al* (1975) from an examination of the coefficients of (4.8). The critical exponent is found to be $\frac{1}{12}$. On the basis of the conjectured form the next four coefficients are:

$$-50\,566\,080u_3^{13}-272\,283\,088u_3^{14}-1\,473\,951\,336u_3^{15}-8\,016\,095\,444u_3^{16}-\ldots. \tag{4.9}$$

5. An order parameter for the simple Ising model

By interchanging the roles of the variables u_2 and u_3 in the preceding section we may define an order parameter for the simple Ising model,

$$P_3 = -\frac{1}{2}\left(\frac{\partial F}{\partial J_3}\right)_{J_3=0} = 1-2\left(u_3\frac{\partial L}{\partial u_3}\right)_{u_3=1} \tag{5.1}$$

which may be regarded as a form of spontaneous triangular polarization. To evaluate the expansion coefficients we require the ferromagnetic polynomials $\psi_n(\mu)$ for the simple model with a modified argument: $\psi_n(u_3)$ defined by interchanging u_2 and u_3 in (4.7). This requires a detailed examination of the configurations that contribute to the u_2-*grouping*; this grouping which as has already been emphasized is not the same as the u_3-*grouping*, has been given by Sykes *et al* (1973) through u_2^{16} and extended by Sykes *et al* (1975a, b) through u_2^{21}. From an analysis of the contributing configurations we have obtained the polynomials $\psi_n(u_3)$ through $n = 21$ and we list them in the appendix. Using (5.10) we have derived the expansion:

$$\begin{aligned} P_3 = {} & 1-6u_2^3-24u_2^5+22u_2^6-126u_2^7+192u_2^8-848u_2^9+1\,440u_2^{10}-6\,258u_2^{11} \\ & +10\,882u_2^{12}-47\,928u_2^{13}+84\,318u_2^{14}-375\,326u_2^{15}+667\,248u_2^{16} \\ & -2\,988\,252u_2^{17}+5\,366\,000u_2^{18}-24\,106\,638u_2^{19}+43\,695\,984u_2^{20} \\ & -196\,565\,300u_2^{21}+\ldots. \end{aligned} \qquad (5.2)$$

Numerical studies of the coefficients indicate a critical exponent of $\frac{1}{8}$. This conclusion has recently been confirmed by Baxter (1975) who has calculated the exact expression for P_3.

6. Summary and conclusions

We have shown that the linkage rule for the pure triplet model can be transformed by using Euler's law of the edges; the power of the temperature variable u_3 corresponding to any configuration is then found to depend only on the number of spins and a discriminant of the configuration. This discriminant, defined as $(c-h)$ for any configuration with c components and h holes, also arises in the derivation of series expansions for the simple Ising model on the triangular lattice by the code method (Sykes *et al* 1975a, b); we have therefore been able to exploit data derived originally for the simple model and so obtain series of useful length for the triplet model.

A study of the expansions derived for certain order parameters ((4.1) and (4.5)) has made it possible to conjecture their exact form (Baxter *et al* 1975). It is interesting to notice that the critical indices for (4.1) and (4.5) for the pure triplet model are both $\frac{1}{12}$ while the index for (5.1) is $\frac{1}{8}$ and therefore identical with the critical index for the magnetization of the simple Ising model.

Acknowledgments

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Appendix

Polynomials Ψ and ψ for mixed model ($H = 0$)

$$\ln \Lambda = \sum_n \Psi_n(u_2)u_3^n = \sum_n \psi_n(u_3)u_2^n$$

$\Psi_3 = u_2^3$

$\Psi_4 = 3u_2^5$

$\Psi_5 = 2u_2^6+9u_2^7$

$\Psi_6 = -3\frac{1}{2}u_2^6+3u_2^7+12u_2^8+29u_2^9+u_2^{12}$

$$\Psi_7 = -24u_2^8+21u_2^9+60u_2^{10}+99u_2^{11}+6u_2^{13}+6u_2^{14}$$

$$\Psi_8 = -10u_2^9-136\tfrac{1}{2}u_2^{10}+129u_2^{11}+280u_2^{12}+348u_2^{13}+27u_2^{14}+36u_2^{15}+36u_2^{16}+3u_2^{19}$$

$$\Psi_9 = 20\tfrac{1}{3}u_2^9-12u_2^{10}-171u_2^{11}-656u_2^{12}+726u_2^{13}+1\,248u_2^{14}+1\,327u_2^{15}+186u_2^{16} +228u_2^{17}+190u_2^{18}+9u_2^{19}+18u_2^{20}+24u_2^{21}+2u_2^{24}$$

$$\Psi_{10} = 6u_2^{10}+261u_2^{11}-313u_2^{12}-1\,428u_2^{13}-2\,860\tfrac{1}{2}u_2^{14}+3\,807u_2^{15}+5\,577u_2^{16}+5\,118u_2^{17} +1\,263u_2^{18}+1\,281u_2^{19}+1\,017u_2^{20}+147u_2^{21}+156u_2^{22}+171u_2^{23}+6u_2^{24}+12u_2^{25} +27u_2^{26}+3u_2^{29}$$

$$\Psi_{11} = 27u_2^{11}+160u_2^{12}+1\,950u_2^{13}-3\,306u_2^{14}-9\,423u_2^{15}-11\,556u_2^{16}+19\,023u_2^{17} +24\,408_2^{18}+20\,730u_2^{19}+7\,554u_2^{20}+7\,221u_2^{21}+5\,292u_2^{22}+1\,226u_2^{23} +1\,202u_2^{24}+1\,104u_2^{25}+138u_2^{26}+180u_2^{27}+222_2^{28}+12u_2^{29}+18u_2^{30} +42u_2^{31}+6u_2^{34}$$

$$\Psi_{12} = 3u_2^{11}-63\tfrac{3}{4}u_2^{12}+312u_2^{13}+2\,332\tfrac{1}{2}u_2^{14}+11\,688u_2^{15}-26\,742u_2^{16}-54\,501u_2^{17} -44\,431\tfrac{1}{2}u_2^{18}+91\,008u_2^{19}+107\,112u_2^{20}+86\,849u_2^{21}+43\,674u_2^{22}+39\,090u_2^{23} +27\,296\tfrac{1}{2}u_2^{24}+9\,564u_2^{25}+8\,355u_2^{26}+6\,824u_2^{27}+1\,641u_2^{28}+1\,605u_2^{29}+1\,677u_2^{3} +231u_2^{31}+288u_2^{32}+392_2^{33}+30u_2^{34}+36u_2^{35}+87u_2^{36}+14u_2^{39}+u_2^{42}$$

$$\psi_3 = u_3^3$$

$$\psi_4 = 0$$

$$\psi_5 = 3u_3^4$$

$$\psi_6 = 2u_3^5-3\tfrac{1}{2}u_3^6$$

$$\psi_7 = 9u_3^5+3u_3^6$$

$$\psi_8 = 12u_3^6-24u_3^7$$

$$\psi_9 = 29u_3^6+21u_3^7-10u_3^8+20\tfrac{1}{3}u_3^9$$

$$\psi_{10} = 60u_3^7-136\tfrac{1}{2}u_3^8-12u_3^9+6u_3^{10}$$

$$\psi_{11} = 99u_3^7+129u_3^8-171u_3^9+261u_3^{10}+27u_3^{11}+3u_3^{12}$$

$$\psi_{12} = u_3^6+280u_3^8-656u_3^9-313u_3^{10}+160u_3^{11}-62\tfrac{3}{4}u_3^{12}+24u_3^{13}+2u_3^{14}$$

$$\psi_{13} = 6u_3^7+348u_3^8+726u_3^9-1\,428u_3^{10}+1\,950u_3^{11}+312u_3^{12}+123u_3^{13}+117u_3^{14}+27u_3^{15} +3u_3^{16}$$

$$\psi_{14} = 6u_3^7+27u_3^8+1\,248u_3^9-2\,860\tfrac{1}{2}u_3^{10}-3\,306u_3^{11}+2\,332\tfrac{1}{2}u_3^{12}-2\,034u_3^{13}+81u_3^{14} +342u_3^{15}+168u_3^{16}+42u_3^{17}+6u_3^{18}$$

$$\psi_{15} = 36u_3^8+1\,327u_3^9+3\,807u_3^{10}-9\,423u_3^{11}+11\,688u_3^{12}+4\,522u_3^{13}-425u_3^{14} +1\,209\tfrac{1}{5}u_3^{15}+696u_3^{16}+675u_3^{17}+290u_3^{18}+87u_3^{19}+14u_3^{20}+u_3^{21}$$

$$\begin{aligned}\psi_{16} = {} & 36u_3^8+186u_3^9+5\,577u_3^{10}+11\,556u_3^{11}-26\,742u_3^{12}+24\,714u_3^{13}-24\,306u_3^{14}\\ & -2\,196u_3^{15}+1\,206u_3^{16}+858u_3^{17}+1\,839u_3^{18}+1\,278u_3^{19}+606u_3^{20}+198u_3^{21}\\ & +42u_3^{22}+6u_3^{23}\end{aligned}$$

$$\begin{aligned}\psi_{17} = {} & 228u_3^9+5\,118u_3^{10}+19\,023u_3^{11}-54\,501u_3^{12}+57\,567u_3^{13}+59\,106u_3^{14}-25\,821u_3^{15}\\ & +19\,860u_3^{16}+5\,511u_3^{17}+1\,677u_3^{18}+3\,669u_3^{19}+4\,068u_3^{20}+2\,745u_3^{21}\\ & +1\,401u_3^{22}+507u_3^{23}+147u_3^{24}+27u_3^{25}+3u_3^{26}\end{aligned}$$

$$\begin{aligned}\psi_{18} = {} & 190u_3^9+1\,263u_3^{10}+24\,408u_3^{11}-44\,431\tfrac{1}{2}u_3^{12}-185\,554u_3^{13}+203\,108\tfrac{1}{2}u_3^{14}\\ & -188\,877\tfrac{1}{3}u_3^{15}-59\,612u_3^{16}+1\,514u_3^{17}-1\,233\tfrac{1}{6}u_3^{18}+6\,346u_3^{19}+5\,646u_3^{20}\\ & +9\,812u_3^{21}+8\,961u_3^{22}+6\,432u_3^{23}+3\,387u_3^{24}+1\,458u_3^{25}+496u_3^{26}+128u_3^{27}\\ & +24u_3^{28}+2u_3^{29}\end{aligned}$$

$$\begin{aligned}\psi_{19} = {} & 3u_3^8+9u_3^9+1\,281u_3^{10}+20\,730u_3^{11}+91\,008u_3^{12}-293\,976u_3^{13}+225\,351u_3^{14}\\ & +606\,705u_3^{15}-374\,034u_3^{16}+274\,230u_3^{17}+37\,959u_3^{18}-4\,446u_3^{19}\\ & +23\,814u_3^{20}+11\,673u_3^{21}+19\,779u_3^{22}+23\,109u_3^{23}+21\,525u_3^{24}+15\,477u_3^{25}\\ & +8\,898u_3^{26}+4\,338u_3^{27}+1\,719u_3^{28}+579u_3^{29}+147u_3^{30}+27u_3^{31}+3u_3^{32}\end{aligned}$$

$$\begin{aligned}\psi_{20} = {} & 18u_3^9+1\,017u_3^{10}+7\,554u_3^{11}+107\,112u_3^{12}-163\,722u_3^{13}-1\,169\,742u_3^{14}\\ & +1\,397\,904u_3^{15}-1\,086\,656\tfrac{1}{4}u_3^{16}-948\,006u_3^{17}+195\,327u_3^{18}-170\,604u_3^{19}\\ & -59\,115u_3^{20}+35\,676u_3^{21}+34\,671_3^{22}+36\,234u_3^{23}+50\,997u_3^{24}+55\,788u_3^{25}\\ & +52\,479u_3^{26}+38\,472u_3^{27}+24\,465u_3^{28}+12\,996u_3^{29}+6\,060u_3^{30}+2\,412u_3^{31}\\ & +798u_3^{32}+216u_3^{33}+42u_3^{34}+6u_3^{35}\end{aligned}$$

$$\begin{aligned}\psi_{21} = {} & 24u_3^9+147u_3^{10}+7\,221u_3^{11}+86\,849u_3^{12}+424\,155u_3^{13}-1\,515\,743u_3^{14}+569\,241u_3^{15}\\ & +5\,178\,485u_3^{16}-3\,769\,998u_3^{17}+2\,654\,610u_3^{18}+654\,652u_3^{19}-32\,840u_3^{20}\\ & +51\,970\tfrac{1}{7}u_3^{21}-3\,059u_3^{22}+105\,368u_3^{23}+75\,365u_3^{24}+109\,843u_3^{25}\\ & +124\,030u_3^{26}+141\,148u_3^{27}+127\,947u_3^{28}+100\,612u_3^{29}+67\,786u_3^{30}\\ & +39\,912u_3^{31}+20\,987u_3^{32}+9\,632u_3^{33}+3\,918u_3^{34}+1\,341u_3^{35}+392u_3^{36}\\ & +87u_3^{37}+14u_3^{38}+u_3^{39}\end{aligned}$$

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